

Determination of the Dynamic ITER Energy Confinement Time Scalings

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Abstract

We derive the differential equation, which is satisfied by the ITER scalings for the dynamic energy confinement time. We show that this differential equation can also be obtained from the differential equation for the energy confinement time, derived from the energy balance equation, when the plasma is near the steady state. We find that the values of the scaling parameters are linked to the second derivative of the power loss, estimated at the steady state. As an example of an application, the solution of the differential equation for the energy confinement time is compared with the profile obtained by solving numerically the balance equations (closed by a transport model) for a concrete Tokamak-plasma.

PACS numbers: 28.52.-s, 28.52.Av

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I. INTRODUCTION

Global scaling expressions for the energy confinement time, τ_E , or the stored energy, W , are powerful tools for predicting the confinement performance of burning plasmas [1], [2], [3]. The fusion performance of ITER is predicted using three different techniques: statistical analysis of the global energy confinement data in the parameters (simple (multivariate) linear regression tools can be used to determine the parameters from a set of data) [4], [5], a dimensionless scaling analysis, based on dimensionless physics parameters [6], [5], [7], and theory-based on transport models and modelling the plasma profiles [8], [9] and [10]. Although the three methods give overlapping predictions for the performance of ITER, the confidence interval of all of the techniques is still quite wide [11]. The Confinement Database and Modelling Expert Group recommended for ITER design the so-called *ITERH* – 98 $P(y, 2)$ confinement scaling [5], [12]:

$$\tau_E^{H98(y,2)} = 0.0562 I_p^{0.93} R^{1.97} \epsilon^{0.58} \kappa^{0.78} B_{0\phi}^{0.15} \bar{n}_e^{0.41} P^{-0.69} M^{0.19} \quad (1)$$

Here, the parameters are the plasma current I_p , the major radius R , the inverse aspect ratio $\epsilon = a/R$ (with a denoting the minor radius of the Tokamak), the elongation κ , the toroidal magnetic field (at the major radius R) $B_{0\phi}$, the central line averaged electron density \bar{n}_e , the loss power P , and the ion mass number M , respectively. The expression (1) is valid for the ELMy H-mode thermal energy confinement time. The 2 log-linear interval was determined to be 20%. By recent analyzing the enlarged *ITERH.DB3* dataset, the practical reliability of the *ITERH* – 98 $(y, 2)$ scaling was confirmed and 2 log-linear interval was reduced to 14% [13]. Tables showing some of the most generally used sets of scaling parameters for the ELMy H-mode and L-mode can be found in Refs [5], [14], [15] and [16].

For stellarators, a similar scaling has been obtained [17], [18]

$$\tau_E = 0.148 R^{0.64} a^{2.33} \bar{n}_e^{0.55} B_{0\phi}^{0.85} \iota^{0.41} P^{-0.61} \quad (2)$$

where $\iota/2\pi$ is the rotational transform (or the field line pitch).

The confinement time is defined as

$$\tau_E = \frac{W_e}{P_{tot} - \dot{W}_e} = \frac{W_e}{P_Q} \quad (3)$$

where W_e , P_Q and P_{tot} are the internal energy, the power loss and the power source, respectively. From Eq.(3) results that when the tokamak is not in the steady state the quantity

τ_E is a time dependent quantity. Hence, τ_E , given by Eqs (1) and (2), is viewed as a time-dependent variable, which depends on a collection of variables *dependent* on time (*e.g.*, \bar{n} , P , etc). The value of τ_E at the *steady state condition* $\dot{\tau}_E = 0$, *attained at some time moment* t_0 , corresponds to the numerical value provided by the database. For example, the point prediction for the thermal energy confinement time in ITER is $(\tau_E, \dot{\tau}_E) = (3.6 \text{ sec}, 0)$.

The main objective of this work is to estimate the energy confinement time, close to the steady state. τ_E at the steady state condition is calculated by using the expression

$$\tau_E^0 = \frac{W_{estat.}}{P_{Qstat.}} \quad (4)$$

where $W_{estat.}$ and $P_{Qstat.}$ are obtained by solving the *stationary* balance equations. An example of calculation can be found in Ref. [19]. To estimate the dynamic confinement time we should solve the evolutive balance equations. However, this is a very complex task. An alternative strategy (which is the one that we shall adopt here) consists in deriving the time differential equation for the energy confinement time, with τ_E^0 , estimated by using Eq. (4), playing the role of the initial condition. We show that τ_E is the solution of a nonlinear differential equation of second order in time, obtained by combining Eq. (3) with the (dynamic) balance equations. The critical fact which makes our approach useful is that in the vicinity of the stationary state, this differential equation depends only on one coefficient which varies *very slowing in time*

$$\begin{cases} \tau_E \ddot{\tau}_E - \dot{\tau}_E^2 = \chi(t) \tau_E^2 \\ \tau_E^0 = \frac{W_{estat.}}{P_{Qstat.}} \quad ; \quad \dot{\tau}_E = 0 \end{cases} \quad (5)$$

where $\chi(t) \simeq \chi_0(t - t_0)$, and χ_0 is a numerical coefficient estimated at the steady state. Hence, at the "leading order", all of the dependence on the machine is reduced to just a number, χ_0 , which can be determined. This is the real advantage of this approach. As an example of calculation, we have considered the simplest case of IGNITOR-plasmas. In this case, we solved the time differential equation for τ_E where the parameters (*i.e.*, the initial condition as well as the coefficient appearing in the differential equation) have been estimated at the steady state. The solution of this equation is in agreement with the one obtained by solving numerically the dynamic balance equations, with the aid of a transport model [20].

In this work, we shall also justify the dynamic scaling laws, like

$$\tau_E = C I_p^{\alpha_1} \bar{n}_e^{\alpha_2} P^{a_3} M^{\alpha_4}, \quad (6)$$

where C is a constant and M is the effective mass, respectively (note that when the plasma is a mixture, due to the dependence of particle transport properties on particle mass and charge, M is also time dependent). In particular, we shall prove that the dynamic expression for the energy confinement time, like Eq. (6), is solution of the differential equation for τ_E , which can be obtained by combining Eq. (3) with the energy balance equation.

The paper is organized as follows. In Section (II), we show that Eqs (6) satisfy a nonlinear differential equation of the second order in time, taking into account the (experimentally established) slow variation in time of the coefficient entering in this equation. Successively, we show that this equation can also be derived from the energy balance equation, combined with definition (3). This will allow a linking of the scaling coefficients with the (measurable) second time derivatives of the heat power loss, which at the leading order may also be estimated at the stationary state. These tasks will be accomplished in the Section (III). As an example of an application, in the Section (IV), we compare the solution obtained by solving the differential equation for the energy confinement time with the numerical simulations obtained using the code JETTO [20], for the specific case of IGNITOR-plasmas. Concluding remarks can be found in Section (V).

II. DIFFERENTIAL EQUATION SATISFIED BY THE ITER SCALINGS

The expression for the energy confinement time, obtained by scaling laws, raises several questions. Firstly, Eq. (1) applies quite well to a large number of Tokamaks (ASDEX, JET, DIII-D, ALCATOR C-Mod, COMPASS, etc.) and it is currently used for predicting the energy confinement time for Tokamaks, which are presently in construction (ITER) or will be constructed in the future (DEMO). Hence, the first objective of this work is to understand *the main reason for such a "universal" validity*. Secondly, it is legitimate to ask *"where does this expression originate from ?"*. More concretely, *"Is it possible to determine the (minimal) differential equation which is satisfied by expression (6) ?"*. In case of a positive answer, *"Is it possible to re-obtain this (minimal) differential equation from the balance equations and, in particular, from the energy balance equations ?"*. Finally, *"How can we estimate the values*

of the scaling coefficients α_i ?". In this Section, we shall determine the (minimal) differential equation satisfied by Eqs (6). In the next Section we shall prove that, near the stationary state, this differential equation can be re-obtained from the energy balance equation.

The equations of one-dimensional plasma dynamics, in toroidal geometry, assuming the validity of the standard model, can be brought into the form (see, for example, [21])

$$\begin{aligned} \frac{\partial n_e}{\partial t} &= -\frac{1}{r} \frac{\partial}{\partial r} \left(r \langle \gamma_r^e \rangle \right) \\ \frac{3}{2} \frac{\partial p}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left[r \left(\langle q_e \rangle + \langle q_i \rangle + \frac{5}{2} (1 + Z^{-1}) T_e \langle \gamma_r^e \rangle \right) \right] &= \\ &= \frac{c}{4\pi} \frac{E_0 B_{0\phi}}{Rr} \frac{\partial}{\partial r} \left(\frac{r^2}{q(r)} \right) + S_{gain-loss} \end{aligned} \quad (7)$$

with r and $q(r)$ denoting the radial coordinate and the safety factor, respectively. p , n_e , T_e and Z are the total plasma pressure, the electron density, the electron temperature and the ion charge number, respectively. Here, $\langle \dots \rangle$ denotes the surface-average operation. $\langle q_\zeta \rangle$ and $\langle \gamma_r^e \rangle$ are the averaged radial heat flux of species ζ ($\zeta = e$ for electrons and $\zeta = i$ for ions) and the averaged electron flux, respectively. c and E_0 are light speed and the external electric field, respectively, and $S_{gain-loss}$ is the source term, *i.e.* the loss and energy gain. Eq. (7) must be completed with the transport equations, *i.e.* with the thermodynamic flux-force relations, in order to close the plasma dynamical equations. The 0 - D power balance equation is now derived as follows. Eq. (7) is integrated over the volume of the plasma and then divided by the plasma volume V . We obtain

$$\begin{cases} \dot{N}_e = -\Gamma \\ \dot{W}_e + P_Q = P_{tot} \end{cases} \quad (8)$$

with

$$\begin{aligned} N_e &\equiv V^{-1} \int n_e dV \quad ; \quad \Gamma \equiv V^{-1} \int \frac{1}{r} \frac{\partial}{\partial r} \left(r \langle \gamma_r^e \rangle \right) dV \\ W_e &\equiv \frac{3}{2} V^{-1} \int p dV \quad ; \quad P_{tot} \equiv V^{-1} \int \left[\frac{c}{4\pi} \frac{E_0 B_{0\phi}}{Rr} \frac{\partial}{\partial r} \left(\frac{r^2}{q(r)} \right) + S_{gain-loss} \right] dV \\ P_Q &\equiv V^{-1} \int \left(\frac{1}{r} \frac{\partial}{\partial r} \left[r \left(\langle q_e \rangle + \langle q_i \rangle + \frac{5}{2} (1 + Z^{-1}) T_e \langle \gamma_r^e \rangle \right) \right] \right) dV \end{aligned} \quad (9)$$

where the "dot" over the variables stands for the (total) time derivative (d/dt).

The energy confinement time is defined as

$$\tau_E = \frac{W_e}{P_{tot} - \dot{W}_e} = \frac{W_e}{P_Q} \quad (10)$$

From definition (10), we find

$$\dot{\tau}_E P_Q + \tau_E \dot{P}_Q - \dot{W}_e = 0 \quad (11)$$

Note that the stationary state is reached when $P_Q = P_{tot}$. Hence, at the steady state (corresponding to $t = t_0$) we have

$$\dot{W}_e|_{t=t_0} \equiv \dot{W}_e^0 = 0 \quad (12)$$

At the steady state, we find

$$\tau_E(t_0) \equiv \tau_E^0 = \frac{W_e^0}{P_{tot}^0} \quad ; \quad \left. \frac{d\tau_E}{dt} \right|_{t=t_0} \equiv \dot{\tau}_E^0 = 0 \quad (13)$$

where W_e^0 and P_{tot}^0 indicate the values of W_e and P_{tot} , estimated at the steady state, respectively.

Eq. (6) may be re-written in the generic form:

$$\tau_E = C X_1^{\alpha_1} X_2^{\alpha_2} \dots X_n^{\alpha_n} \quad (14)$$

where X_1, X_2, \dots are a positive and independent system of variables X_i , and α_i the *scaling parameters*, respectively. For simplicity, we firstly suppose that in Eq. (14) all the variables X_i are time-dependent. The case whereby X_i is a collection of variables dependent on time, as well as variables not-dependent on time, will be treated in the following sub-Section *Analysis in the Physics Variables*. Note that C is a (dimensional) constant satisfying the condition

$$C = \tau_E^0 X_1(t_0)^{-\alpha_1} X_2(t_0)^{-\alpha_2} \dots X_n(t_0)^{-\alpha_n} \quad (15)$$

Unless stated otherwise, in the sequel we shall adopt the summation convention on the repeated indexes. By taking the logarithm of Eq. (14) we find

$$y = \log C + \alpha_1 \xi_1 + \alpha_2 \xi_2 + \dots \alpha_n \xi_n \quad (16)$$

with $y \equiv \log \tau_E$ and $\xi_i \equiv \log X_i$ (with $i = 1, \dots, n$). The first and the second derivatives of y , with respect to variable ξ_i , read respectively

$$\frac{\partial y}{\partial \xi_i} = \alpha_i \quad ; \quad \frac{\partial^2 y}{\partial \xi_i \partial \xi_j} = 0 \quad (17)$$

In terms of variable τ_E , instead of y , we get

$$\frac{\partial \tau_E}{\partial \xi_i} = \tau_E \alpha_i \quad ; \quad \tau_E \frac{\partial^2 \tau_E}{\partial \xi_i \partial \xi_j} - \frac{\partial \tau_E}{\partial \xi_i} \frac{\partial \tau_E}{\partial \xi_j} = 0 \quad (18)$$

The differential equation with respect to time is easily obtained by tacking into account the identities

$$\dot{\tau}_E = \frac{\partial \tau_E}{\partial \xi_i} \dot{\xi}_i = \tau_E \alpha_i \dot{\xi}_i \quad ; \quad \frac{\partial^2 \tau_E}{\partial \xi_i \partial \xi_j} \dot{\xi}_i \dot{\xi}_j = \ddot{\tau}_E - \frac{\partial \tau_E}{\partial \xi_i} \ddot{\xi}_i = \ddot{\tau}_E - \tau_E \alpha_i \ddot{\xi}_i \quad (19)$$

By multiplying the second equation of Eqs (18) by $\dot{\xi}_i \dot{\xi}_j$ and by summing over indexes, we finally obtain *the differential equation satisfied by the ITER scaling laws*

$$\tau_E \ddot{\tau}_E - \dot{\tau}_E^2 = \left(\sum_{i=1}^n \alpha_i \ddot{\xi}_i(t) \right) \tau_E^2 \quad (20)$$

Eq. (20) should be solved with the initial conditions (13):

$$\begin{cases} \tau_E \ddot{\tau}_E - \dot{\tau}_E^2 = \chi(t) \tau_E^2 \\ \tau_E^0 = \frac{W_e^0}{P_{tot}^0} \quad ; \quad \dot{\tau}_E^0 = 0 \end{cases} \quad (21)$$

with $\chi(t) \equiv \left(\sum_{i=1}^n \alpha_i \ddot{\xi}_i(t) \right)$. We have derived two differential equations for the time derivatives of τ , the first equation of Eq. (19) which is first order and also Eq. (20) which is second order. It may appear hopeless to solve these equations, as they depend on $\alpha_i \dot{\xi}_i(t)$ and $\chi(t) = \alpha_i \ddot{\xi}_i(t)$ respectively, which in turn depend on the full dynamics of the system. The critical fact which makes our approach useful is that the second time derivatives of the logarithm of X_i are generally weakly dependent on time. As a result, one may approximate $\chi(t)$ to be a constant, χ_0 . In this sense, all of the dependence on the machine is reduced to just a number, which can be determined. The evolution of τ_E can then be obtained uniquely by integrating Eq. (20) with the initial conditions (13). Such an approach would not work for the first equation of Eq. (19) as $\alpha_i \dot{\xi}_i(t)$ depends strongly on time, indeed it vanishes at the initial stationary state and then becomes nonzero as the state evolves.

It is not difficult to check that the nonlinear equation (21) is the "minimal" differential equation, in the sense that Eq. (21) admits one, and only one, solution (*i.e.*, the nonlinear differential equation (21) *does not generate* additional solutions).

It may appear hopeless to solve Eq. (21), as it depends on the coefficient $\chi = \left(\sum_{i=1}^n \alpha_i \ddot{\log} X_i \right)$, which in turn depend on the full dynamics of the system. The critical fact which makes our approach useful is that the second time derivatives of the logarithm of X_i are generally weakly time-dependent. In all the cases examined by the authors, $\chi(t)$

is very well approximated (numerically) by a linear function in time

$$\chi(t) \simeq \chi_0(t - t_0) \quad \text{with} \quad \chi_0 = -\frac{1}{t_0} \sum_{i=1}^n \alpha_i \ddot{\xi}_i(t_0) \quad (22)$$

Hence, all of the dependence on the machine is reduced to just a number, χ_0 , which can be *estimated at the steady state*.

III. DIFFERENTIAL EQUATION FOR THE ENERGY CONFINEMENT TIME

The aim of this Section is to obtain the differential equation for the energy confinement time from the balance equations. In analogy with Eq. (21), the coefficients of this differential equation should be expressed only in terms of the internal energy W_e and the total power P_{tot} . To this end, let us reconsider the energy balance equation Eq. (8) and the definition of the energy confinement time, Eq. (10). Taking the derivative of Eq. (11) with respect to time, after a little algebra, we get

$$\tau_E \ddot{\tau}_E - \dot{\tau}_E^2 = -f(t)\tau_E^2 - g(t)\tau_E \dot{\tau}_E \quad (23)$$

with

$$\begin{aligned} f(t) &\equiv \frac{\ddot{P}_{tot} - \ddot{W}_e}{P_{tot} - \dot{W}_e} - \frac{\ddot{W}_e}{W_e} = -\chi(t) \\ g(t) &\equiv \frac{\dot{P}_{tot} - \dot{W}_e}{P_{tot} - \dot{W}_e} + \frac{\dot{W}_e}{W_e} \end{aligned} \quad (24)$$

Note that the dimensions of $f(t)$ and $g(t)$ are $[t]^{-2}$ and $[t]^{-1}$, respectively. Finally, the *differential equation for the energy confinement time* reads

$$\begin{cases} \tau_E \ddot{\tau}_E - \dot{\tau}_E^2 + f(t)\tau_E^2 + g(t)\tau_E \dot{\tau}_E = 0 \\ \tau_E^0 = \frac{W_e^0}{P_{tot}^0} \quad ; \quad \dot{\tau}_E^0 = 0 \end{cases} \quad (25)$$

We might object that the previous equation has the same degree of difficulty as the initial expression, Eq. (10). However, as we shall see more in detail in the next Subsection, the coefficients $g(t)$ and $f(t)$ possess special properties: close to the steady state $g(t)$ tends to vanish and $f(t)$ is a function varying very slowly in time. So, at the leading order, $g(t) \approx 0$ and $f(t)$ may be estimated at the stationary state [see Eq. (22) and the discussion after

Eq. (21)]. This is the real advantage of Eq. (25) with respect to Eq. (10) : *Eq. (25) allows determining the dynamic behaviour of the energy confinement time when the system is close to the steady state, solely by the knowledge of one coefficient estimated at the stationary state.* Moreover, from the previous Section we know that this equation admits one (and only one) solution corresponding to the ITER scalings. A concrete application of Eq. (25) can be found in the Section (IV). Note that Eq. (25) may be re-written in the more convenient form

$$\begin{cases} \dot{\tau}_E = \tau_E y \\ \dot{y} + g(t)y + f(t) = 0 \\ \tau_E(t_0) = \frac{W_e^0}{P_{tot}^0} \quad ; \quad y(t_0) = 0 \end{cases} \quad (26)$$

showing that the differential equation for the energy confinement time may be expressed as two *quasi-decoupled* differential equations of first order in time derivative. The general solution of Eqs (26) may be brought into the form

$$\tau_E(t) = \tau_E^0 \exp \left[- \int_{t_0}^t dx'' \left(\exp \left(- \int_{t_0}^{x''} dx g(x) \right) \left[\int_{t_0}^{x''} dx' f(x') \exp \left(\int_{t_0}^{x'} dx g(x) \right) \right] \right) \right] \quad (27)$$

By taking into account that $f(t) = - \sum_{i=1}^n \alpha_i \ddot{\xi}_i$ (with $\xi_i = \log X_i$), solution (27) generalizes the ITER scaling laws out of the steady state, reducing to Eq. (14) close to the stationary state. Eq. (27) shows that close to the steady state, the leading contribution to the mathematical expression for the energy confinement time is provided by the power laws. However, when we deviate from the steady state, supplementary contributions, which are different from the power ones, may modify the mathematical form of the power laws significantly. Generally, for ITER, these contributions tend to lower the numerical value of the energy confinement time.

• Differential Equation for the Energy Confinement Time Near the Steady State

The term P_{tot} is specified as follows

$$P_{tot} = P_\alpha(T) - P_b(T) + P_{Aux}(r, t) \quad (28)$$

where $P_\alpha(T)$ is the alpha power, $P_b(T)$ is the power radiation loss (Bremsstrahlung) and P_{aux} is the external heating power density supplied to the system (*e.g.* ohmic heating power or external RF), respectively. The alpha power and the Bremsstrahlung power loss depend explicitly on the temperature of the plasma. The auxiliary heating power

is operational during both the transient and steady states. This is the dominant source of external heating power, and it is assumed to be deposited in the plasma with a known profile, independent of p and T . Hence, $P_{Aux} = P_{Aux}(r, t)$. The time derivative of P_{tot} reads

$$\dot{P}_{tot} = \frac{\partial P_\alpha}{\partial T} \dot{T} - \frac{\partial P_b}{\partial T} \dot{T} + \dot{P}_{Aux} \quad (29)$$

At the steady state $\dot{T}(t_0) = 0$ and $\dot{P}_Q(t_0) = \dot{P}_{Aux}(t_0) = 0$. Consequently, from the energy balance equation we find that also $\ddot{W}_e(t_0) = 0$. By taking into account Eqs (12) and (24), we get $g(t) \rightarrow 0$ as the system approaches the steady state. Hence, near the stationary state, we find

$$\begin{cases} \tau_E \ddot{\tau}_E - \dot{\tau}_E^2 \simeq \chi(t) \tau_E^2 \\ \tau_E^0 = \frac{W_e^0}{P_{tot}^0} \quad ; \quad \dot{\tau}_E^0 = 0 \end{cases} \quad (30)$$

with

$$\chi(t) = -\frac{\ddot{P}_{tot} - \ddot{W}_e}{P_{tot}} = \sum_{i=1}^n \alpha_i \ddot{\xi}_i(t) \simeq \chi_0(t - t_0) \quad (31)$$

where Eq. (22) has been used. As shown in the Section (II), Eq. (30) admits one (and only one) solution, corresponding to the ITER scalings Eq. (6). Note that Eq. (31) provides the desired relation between the exponent coefficients α_i and the macroscopic quantities P_{tot} and W_e . If we have n free exponent coefficients α_i , we can set the following n relations

$$\sum_{i=1}^n \alpha_i \ddot{\xi}_i(t_k) = -\frac{\ddot{P}_{tot}(t_k) - \ddot{W}_e(t_k)}{P_{tot}(t_k)} \quad \text{with} \quad k = 0, 1, \dots, n-1 \quad (32)$$

Eq. (32) link the exponent coefficients with variables which, at least in principle, are under the control of the experimental physicist.

• Analysis in the "Physics" Variables

As mentioned, Eqs (1) and (2) are composed by several variables independent of time (*e.g.*, major and minor radii, elongation etc.). In this case, it is more convenient to express the energy confinement time only in terms of the time-dependent variables. Let us suppose that m variables are time-dependent and the remaining $n - m$ not. In this case, the energy confinement time takes the form [see Eq. (15)]

$$\tau_E = \tau_E^0 \left(\frac{X_1^{\alpha_1}}{X_1^{\alpha_1}(t_0)} \right) \left(\frac{X_2^{\alpha_2}}{X_2^{\alpha_2}(t_0)} \right) \dots \left(\frac{X_n^{\alpha_m}}{X_m^{\alpha_m}(t_0)} \right) \quad (33)$$

where, now, the independent variables $X_i^{\alpha_i}(t)/X_i^{\alpha_i}(t_0)$ are dimensionless. Note that in this case variables ξ_i are defined as $\xi_i = \log(X_i/X_i(t_0))$ (no summation convention over the repeated indexes). Of course, this operation reduces the number of independent variables. However, this number may be reduced further if, instead of "engineering variables", the confinement time is expressed in terms of "physics" parameters such as ρ^* (normalized Larmor radius), β (normalized pressure), ν^* (collisionality), etc. Indeed, according to the observation of Kadomtsev, the transport in the plasma core should be fundamentally governed by three physical dimensionless plasma parameters ρ^* , β and ν^* [22]. In this respect, an interesting paper is Ref. [23]. In [23] the authors show that, due to the Kadomtsev constraint, the final expression for the ELMy H-mode thermal confinement time has only one free exponent coefficient, according to the law:

$$\tau_E^{best} = 2\pi \cdot 10^{-3} I_p \epsilon^{-1} n_e^{\alpha_{ne}} P^{\star-(6+8\alpha_n)/15} \quad (34)$$

with P^* denoting the density of the power loss (*i.e.*, $P^* \equiv P/V$). With the choice $\alpha_{ne} = 1/2$, in "physics" variables, scaling (34) goes as $\alpha_{\rho^*} = -1$ (*i.e.* a gyro-Bohm-like scaling), $\alpha_\beta = -0.5$ and $\alpha_{\nu^*} = 0$. This choice may be tested by using Eq. (32) which, in this particular case, reads

$$15\alpha_{ne}\ddot{\log}n_{0e} - (6 + 8\alpha_{ne})\ddot{\log}P_0^* = -15\frac{\ddot{P}_{tot}^0 - \ddot{W}_{0e}}{P_{tot}^0} \quad (35)$$

where Eq. (33) has been taken into account. We find

$$\alpha_{ne} = \frac{6P_{tot}^0\ddot{\log}P_0^* - 15(\ddot{P}_{tot}^0 - \ddot{W}_{0e})}{P_{tot}^0(15\ddot{\log}n_{0e} - 8\ddot{\log}P_0^*)} \quad (36)$$

IV. COMPARISON WITH THE NUMERICAL SIMULATION OF THE BALANCE EQUATIONS FOR AN L-MODE TOKAMAK-PLASMA

As an example application, we consider in this Section the case of one of the simplest L-mode Tokamak-plasma where the evolution of the energy confinement time has been estimated by solving numerically the balance equations, completed with a transport model. In [20] we find the profile of τ_E against time for Ignitor-plasma. The numerical solution has been obtained by using the code JETTO. To compare this profile with the numerical solution of Eq. (21), we should firstly estimate t_0 , τ_E^0 and $\chi_0 = \frac{1}{t_0}\ddot{P}_Q(t_0)/P_{tot}(t_0)$ [see Eq (22)

and (24)]. In [19], we have estimated the values of these parameters for Ignitor subject to ICRH power (*i.e.*, $P_{Aux} = P_{ICRH}$). The scenario is considered where IGNITOR is led to operate in a slightly sub-critical regime by adding a small fraction of 3He to the nominal 50 – 50 Deuterium-Tritium mixture. The difference between power lost and alpha heating is compensated by an additional ICRH power equal to 1.46 MW, which should be able to increase the global plasma temperature via collisions between 3He minority and the background $D - T$ ions. The analytical expression for the ICRH power profiles inside the plasma has been deduced by fitting the numerical results giving an expression for $P_{Aux} = P_{ICRH}(r)$, which is essentially independent of the bulk temperature. Denoting the ICRH power-density as P_{ICRH}^* , we have

$$P_{ICRH}^*(r) = P_{0ICRH}^* \exp[\tilde{\alpha}_2 B(r_{ICRH})/B_{0\phi}] \exp[-(r - r_{ICRH})^2/\Delta] \quad (37)$$

with $P_{0ICRH}^* = 6.59126 \cdot 10^{-6} MW/m^3$, $\tilde{\alpha}_2 = 15.3478$ and $\Delta = 0.0477032$, respectively.

The value of τ_E^0 has been estimated by the expression [19]

$$\tau_E^0 = \frac{12n_e T}{E_\alpha n_e^2 < \sigma v >_{D-T} - C_B n_e^2 T^{1/2} + 4P_{ICRH}^*} \quad (38)$$

with E_α and C_B denoting the energy at which the alpha particles are created (3.5MeV), and the Bremsstrahlung constant, respectively. σ is the reaction cross section giving a measure of the probability of a fusion reaction as a function of the relative velocity of the two reactant nuclei. $< \sigma v >_{D-T}$ provides an average over the distributions of the product of cross section and velocity v . In the core of the plasma we found [19] $\tau_E^0 = 0.43sec$, $t_0 = 3.5sec$ and $\chi_0 = 0.171429sec^{-3}$. Figs (1) reports on the energy confinement time, τ_E , against time for Ignitor-plasmas in the above mentioned conditions. The profiles have been obtained by solving (with the code JETTO) the balance equations and refer to the ITER scalings *ITER97L* (full dots), *ITER97L** (open dots) and *ITER97L* [20]. Fig. (2) shows the solutions of the differential equation for the ITER scalings, Eq. (21), at the three values of (t_0, τ_E^0) : $(t_0, \tau_E^0) = (0.35sec, 0.43sec)$ (*ITER97L* - blue line), $(t_0, \tau_E^0) = (0.35sec, 0.625sec)$ (*ITER97L** - green line) and $(t_0, \tau_E^0) = (0.35sec, 0.825sec)$ (*ITER97L(P_red)* - brown line). Note that in [20] the authors evaluate the ITER scalings by using the reduced power $P_{red} = P_{tot} - P_{RadTot}$, whereas in our work we use P_{tot} , which includes the Bremsstrahlung radiation loss. This may explain the little difference between the numerical [20] and the analytical slopes.

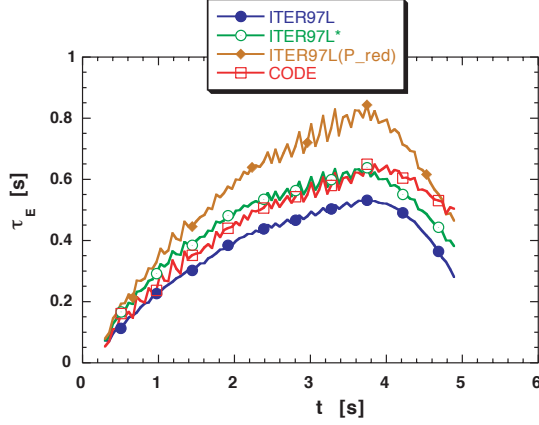


FIG. 1: This is a reproduction of the picture, which appears in [20]. Energy confinement time evolution estimated in [20] by solving with JETTO the balance equations (completed with a transport model): ITER97L scaling (full dots - blue line), ITER97L* scaling (open dots - green line) and ITER97L(P_red) scaling (brown line).

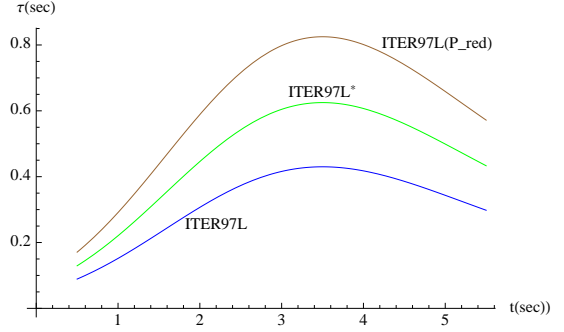


FIG. 2: Solutions of Eq. (21) at the three values of (τ_E^0, t_0) . Blue line: $(t_0, \tau_E^0) = (0.35 \text{ sec}, 0.43 \text{ sec})$ (ITER97L), Green line: $(t_0, \tau_E^0) = (0.35 \text{ sec}, 0.625 \text{ sec})$ (ITER97L*) and Brown line: $(t_0, \tau_E^0) = (0.35 \text{ sec}, 0.825 \text{ sec}) =$ (ITER97L(P_red)).

V. CONCLUSIONS

A large database on plasma energy confinement in Tokamaks can be summarized in single empirical value of τ_E , referred to as the *ITER-scalings*. These expressions are "Universal", in the sense that they apply to a large number of Tokamaks. Scalings are expressed in terms of product of powers of independent variables [see Eq. (14)] and correspond to the *L*-mode as well as the *H*-mode confinements. The recommended scaling for ITER operation remains the *IPB98* scaling law, while this issue is further investigated. In this work we have shown that the ITER scalings satisfy a general non-linear differential equation of second order in time. The value provided by the database for ITER scaling laws, coincides with τ_E^0 , estimated by Eq. (4), with $W_{estat.}$ and $P_{Qstat.}$ evaluated by solving the stationary balance equations. To estimate the dynamic confinement time, we determined the differential equation for τ_E by combining the energy balance equation with definition (3). We found Eqs (25). We have

solved this equation by taking into account that, in vicinity of the steady state, the coefficient $g(t)$ tends to vanish and, at the leading order, $f(t)$ is (almost) a constant independent of time, which may be evaluated at the stationary state. This is the real advantage of the proposed approach: *close to the steady state, the differential equation for the energy confinement time τ_E reduces to*

$$\begin{cases} \tau_E \ddot{\tau}_E - \dot{\tau}_E^2 = \chi(t) \tau_E^2 \\ \tau_E^0 = \frac{W_e^0}{P_{tot}^0} \quad ; \quad \dot{\tau}_E = 0 \end{cases}$$

where "at the leading order" $\chi(t)$ is a numerical constant, which may be estimated at the stationary state. As a result, one may approximate $\chi(t)$ to be a constant, χ_0 or better, by a linear function $\chi(t) = \chi_0(t - t_0)$. In this sense, all of the dependence on the machine is reduced to just a number, χ_0 , which can be estimated at the steady state. Far from the stationary state the differential equation for τ_E contains a nonlinear extra term, which behaves as $\sim \tau_E \dot{\tau}_E$. This extra term tends to modify the mathematical form of the power laws. For ITER, the main effect of this nonlinear extra term is to lower the numerical value of the energy confinement time. The general solution is given by Eq. (27), which reduces to the one admitting the ITER scaling power laws as the system approaches the steady state. We have also seen that the scaling coefficients may be linked to the variables which, at least in principle, are under the control of the experimental physicist. The validity of our approach has been tested by analyzing a concrete example of Tokamak-plasma where the profile of the energy confinement time has been previously determined by solving the balance equations (with the *auxilium* of a transport model). The solution of the differential equation for the ITER scaling is in a fairly agreement with the numerical finding.

VI. ACKNOWLEDGMENTS

JE is supported by NSFC MianShang grant 11375201.

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